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Deformation quantization of compact Kähler manifolds via Berezin-Toeplitz operators

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Abstract

This talk reports on results on the deformation quantization (star products) and on approximative operator representations for quantizable compact Kähler manifolds obtained via Berezin-Toeplitz operators. After choosing a holomorphic quantum line bundle the Berezin-Toeplitz operator associated to a differentiable function on the manifold is the operator defined by multiplying global holomorphic sections of the line bundle with this function and projecting the differentiable section back to the subspace of holomorphic sections. The results were obtained in (respectively based on) joint work with M. Bordemann and E. Meinrenken.

1 The set-up

Let (M, ω) be a compact (complex) Kähler manifold. It should be considered as phase space manifold M with symplectic form given by the Kähler form ω. Note that by the assumed compactness the "free phase space" is not included here. Nevertheless there exist important examples of such compact situations. E.g. they could appear as phase space manifolds for certain constrained systems and they could appear after dividing out certain symmetry group action in a noncompact situation. They also play a role in the quantization of 2-dimensional conformal field theory. Here the space which has to be quantized is the compactified moduli space of stable holomorphic vector bundles (with additional structures) on a Riemann surface.

Using the Kähler form one can assign to every differentiable function[1] f its Hamiltonian vector field X_f and to every pair of functions f and g its Poisson bracket:

$$\omega(X_f, \cdot) = df(\cdot), \qquad \{ f, g \} := \omega(X_f, X_g) \ . \tag{1}$$

With the Poisson bracket the algebra of differentiable functions $C^\infty(M)$ becomes a Poisson algebra $\mathcal{P}(M)$. Assume (M, ω) to be quantizable. This says there exists an associated quantum line bundle (L, h, ∇). Here L is a holomorphic line bundle, h a Hermitian metric on L and ∇ a connection compatible with the complex structure

[1]Differentiable always means arbitrary often differentiable.

and the Hermitian metric. With respect to a local complex coordinate patch and a local holomorphic frame of the bundle the metric h is given by a function $\hat{h}$ and the connection as $(d = \partial + \overline{\partial})$

$$\nabla_{|} = \partial + \partial \log \hat{h} + \overline{\partial} \ . \tag{2}$$

The quantization condition says that the curvature F of the line bundle and the Kähler form ω of the manifold are related as

$$F(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = -\,\mathrm{i}\,\omega(X,Y) \ . \tag{3}$$

In terms of the metric h this can be written as $\mathrm{i}\,\overline{\partial}\partial \log \hat{h} = \omega$. The quantization condition implies that L is a positive (or ample) line bundle. By the Kodaira embedding theorem a certain tensor power of L is very ample, i.e. its global sections can be used to embed the phase space manifold M into projective space (with dimension given by the Riemann-Roch formula). Hence, quantizable compact Kähler manifolds are projective algebraic manifolds (and vice versa, see below).

Example 1. The Riemann sphere, the complex projective line, $\mathbb{P}(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \cong S^2$. With respect to the quasi-global coordinate z the form ω can be given as

$$\omega = \frac{\mathrm{i}}{(1 + z\overline{z})^2} dz \wedge d\overline{z} \ . \tag{4}$$

The quantum line bundle L is the hyperplane bundle. For the Poisson bracket one obtains

$$\{f, g\} = \mathrm{i}\,(1 + z\overline{z})^2 \left(\frac{\partial f}{\partial \overline{z}} \cdot \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial \overline{z}} \right) \ . \tag{5}$$

Example 2. The (complex-) one dimensional torus M. Up to isomorphy it can be given as $M \cong \mathbb{C}/\Gamma_\tau$, where $\Gamma_\tau := \{ n + m\tau \mid n, m \in \mathbb{Z} \}$ is a lattice with $\mathrm{Im}\tau > 0$. As Kähler form we take

$$\omega = \frac{\mathrm{i}\,\pi}{\mathrm{Im}\,\tau} dz \wedge d\overline{z} \ , \tag{6}$$

with respect to the coordinate z on the covering space $\mathbb{C}$. The corresponding quantum line bundle is the theta line bundle of degree 1, i.e. the bundle whose global sections are the scalar multiples of the Riemann theta function.

Example 3. The complex projective space $\mathbb{P}^n(\mathbb{C})$ with the Fubini-Study fundamental form as Kähler form. The associated quantum bundle is the hyperplane bundle. By restricting the form and the bundle on complex submanifolds (which are automatically algebraic) all projective manifolds are quantizable. As remarked above every quantizable compact Kähler manifold is projective. Nevertheless the embedding into projective space does not necessarily respect the Kähler form.

For the space of (arbitrary) global sections of L a scalar product is defined by

$$\langle s_1, s_2 \rangle = \int_M h(s_1, s_2)(x)\Omega(x), \qquad \Omega = \frac{1}{n!} \underbrace{\omega \wedge \omega \cdots \wedge \omega}_{\dim_{\mathbb{C}} M \text{ times}} \ . \tag{7}$$

2 Approximation results for Toeplitz operators

Let $\Gamma_\infty(M,L)$ be the (infinite-dimensional) space of global differentiable sections of the line bundle L, $\mathrm{L}^2(M,L)$ its L^2-completion with respect to the scalar product (7), and $\Gamma_{hol}(M,L)$ the finite-dimensional subspace of global holomorphic sections. Denote by $\Pi : \mathrm{L}^2(M,L) \to \Gamma_{hol}(M,L)$ the projection. The Toeplitz operator associated to the function $f \in \mathcal{P}(M)$ is defined as the composition

$$T_f : \Gamma_{hol}(M,L) \xrightarrow{M_f} \Gamma_\infty(M,L) \xrightarrow{\Pi} \Gamma_{hol}(M,L), \quad s \mapsto f\cdot s \mapsto \Pi(f\cdot s) =: T_f(s)\ . \tag{8}$$

Here M_f denotes the multiplication of the sections by the function f.

Due to the finite-dimensionality of $\Gamma_{hol}(M,L)$ information gets lost. To recover this information we have to do everything for each tensor power $L^m = L^{\otimes m}$ of L. This gives us $(L^m, h^{(m)}, \nabla^{(m)})$ and the Toeplitz operators $T_f^{(m)}$. The assignment

$$T^{(m)} : \mathcal{P}(M) \to \mathrm{End}(\Gamma_{hol}(M,L^m)), \qquad f \mapsto T_f^{(m)} \tag{9}$$

is called *Berezin-Toeplitz quantization* (of level m). The theorems below will justify the use of the term "quantization".

Remark 1. If one switches to the notation $\hbar = 1/m$ then the limit $m \to \infty$ corresponds to $\hbar \to 0$ as "classical limit".

Remark 2. By a result of Tuynman [7] suitable reinterpreted (see [2] for a coordinate independent proof) one obtains the relation $Q_f^{(m)} = \mathrm{i}\, T^{(m)}_{f-\frac{1}{2m}\Delta f}$, where $Q^{(m)}$ is the more well-known operator of geometric quantization with respect to the prequantum operator $P_f^{(m)} = -\nabla^{(m)}_{X_f^{(m)}} + \mathrm{i}\, f\cdot id$ and with Kähler polarization.

Remark 3. In the example of 2-dimensional conformal field theory mentioned at the beginning, there exists a natural quantum line bundle L. The spaces $\Gamma_{hol}(M,L^m)$ are the *Verlinde spaces*.

Assume L to be already very ample. By a rescaling of the Kähler form this always can be achieved. The following two theorems are proved in a joint paper with M. Bordemann and E. Meinrenken [3].

Theorem 1. *For every $f \in C^\infty(M)$ we have $||T_f^{(m)}|| \le ||f||_\infty$ and*

$$\lim_{m\to\infty} ||T_f^{(m)}|| = ||f||_\infty\ . \tag{10}$$

Here $||f||_\infty$ is the sup-norm of f on M and $||T_f^{(m)}||$ is the operator norm with respect to the scalar product (7) on $\Gamma_{hol}(M,L^m)$.

Theorem 2. *For every $f,g \in C^\infty(M)$ we have*

$$||m\,\mathrm{i}\,[T_f^{(m)},T_g^{(m)}] - T^{(m)}_{\{f,g\}}|| \quad = \quad O(\frac{1}{m}) \quad as \quad m\to\infty\ . \tag{11}$$

By interpreting $1/m$ as $\hbar$ we see that up to order $\hbar^2$ the quantization prescription *"Replace the classical Poisson bracket by the (rescaled) commutator of the operators"* is fulfilled. We obtain an approximative operator representation.

3 Deformation quantization (star products)

By a refinement of the methods for the proofs of the above theorems it is possible to construct a star product, resp. a deformation quantization for $\mathcal{P}(M)$. Let me recall the definition of a star product. Let $\mathcal{A} = C^\infty(M)[[\hbar]]$ be the algebra of formal power series in the variable $\hbar$ over the algebra $C^\infty(M)$. A product $\star$ on $\mathcal{A}$ is called a (formal) star product if it is an associative $\mathbb{C}[[\hbar]]$-linear product such that

1. $\mathcal{A}/\hbar\mathcal{A} \cong C^\infty(M)$, i.e. $\quad f \star g \bmod \hbar = f \cdot g$,
2. $\dfrac{1}{\hbar}(f \star g - g \star f) \bmod \hbar = -\mathrm{i}\,\{f, g\}$.

We can always write

$$f \star g = \sum_{j=0}^{\infty} C_j(f,g)\hbar^j \quad \text{with} \quad C_j(f,g) \in C^\infty(M)\ . \tag{12}$$

The conditions 1. and 2. can be reformulated as

$$C_0(f,g) = f \cdot g, \qquad C_1(f,g) - C_1(g,f) = -\mathrm{i}\,\{f,g\}\ . \tag{13}$$

Theorem 3. *There exists a unique (formal) star product on $C^\infty(M)$*

$$f \star g := \sum_{j=0}^{\infty} \hbar^j C_j(f,g), \quad C_j(f,g) \in C^\infty(M), \tag{14}$$

in such a way that for $f, g \in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have

$$||T_f^{(m)} T_g^{(m)} - \sum_{0 \le j < N} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)}|| = K_N(f,g) \left(\frac{1}{m}\right)^N \tag{15}$$

for $m \to \infty$, with suitable constants $K_N(f,g)$.

Again this is joint work with M. Bordemann and E. Meinrenken. The proof can be found in [6]. Also, it will appear in a forthcoming paper.

4 Some additional remarks

Time and space does not permit me to give even a rough sketch of the proof. Let me only point out the following steps. We use the embedding of M via the sections of L (which is now assumed to be very ample) into projective space. For the embedded situation we have an associated Toeplitz structure, the generalized Toeplitz operators (of arbitrary orders) and their symbol calculus as introduced by Boutet de Monvel and Guillemin [4]. The above defined $T_f^{(m)}$ can be considered as the "modes" (with

respect to an S^1-action) of a global Toeplitz operator $\tilde{T}_f$. The symbol calculus gives bounds for their norms.

With the help of the above theorems it is possible to show [3] the conjecture of Bordemann, Hoppe, Schaller and Schlichenmaier [2] that for quantizable Kähler manifolds the Poisson algebra is an $\mathfrak{u}(d(N))$, $N \to \infty$ quasi-limit with a strictly increasing integer valued function $d(N)$ given by the Riemann-Roch formula. For the definition of quasi-limits I refer to the above mentioned article.

References to related works can be found in [5]. Here I only want to quote some names: F.A. Berezin, B.V. Fedosov, J.H. Rawnsley, M. Cahen, S. Gutt, S. Klimek, A. Lesniewski, L.A. Coburn,... Note also that the Berezin-Toeplitz quantization can be considered in the frame-work of "prime quantization" as introduced by S.T. Ali and H.D. Doebner.

The relation of Berezin-Toeplitz quantization to Berezin's coherent states [1] and symbols (in the coordinate independent formulation due to Rawnsley) is studied in [6]. In particular, it is shown that for compact Kähler manifolds the Toeplitz map $f \mapsto T_f$ and Berezin's covariant symbol map $B \mapsto \sigma(B)$ are adjoint maps with respect to the Hilbert-Schmidt scalar product for the operators and the Liouville measure on M which is modified by the ϵ-function of Rawnsley for the functions.

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